Generalized Self-Scalable and Self-Similar Fractal Arrays

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Potential Benefits of Fractal Arrays

- Multiband / Broadband characteristics
- Ability to exploit recursive nature of fractals to develop rapid beamforming algorithms
- Ability to develop schemes for low sidelobe designs
- Provides a systematic approach to thinning
- Provides efficient design strategies for large planar arrays
- Require a minimal amount of switching when implemented as reconfigurable apertures
Pattern Multiplication Theorem for Self-Scalable and Self-Similar Arrays

\[ AF_P(\Psi) = \prod_{p=1}^{P} GA(\delta^{p-1}\Psi) \]

Where

\[ GA(\Psi) = \text{array factor of generating array} \]

\[ \delta = \text{scaling or expansion factor} \]

Circular Ring Array Generator

\[ GA(\theta, \phi) = \sum_{m=1}^{M} \sum_{n=1}^{N_m} I_{mn} e^{j\psi_{mn}(\theta, \phi)} \]

where

\[ \psi_{mn}(\theta, \phi) = k r_m \sin \theta \cos(\phi - \phi_{mn}) + \alpha_{mn} \]

\[ k = \frac{2\pi}{\lambda} \]

\[ M = \text{total number of concentric rings} \]

\[ N_m = \text{total number of elements on the } m^{th} \text{ ring} \]

\[ r_m = \text{radius of the } m^{th} \text{ ring} \]

\[ I_{mn} = \text{excitation current amplitude of the } n^{th} \text{ element on the } m^{th} \text{ ring located at } \phi = \phi_{mn} \]

\[ \alpha_{mn} = \text{excitation current phase of the } n^{th} \text{ element on the } m^{th} \text{ ring located at } \phi = \phi_{mn} \]
The Self-Scalable/Self-Similar Array Factor Corresponding to a Circular Ring Array Generator

\[ GA(\theta, \phi) = \sum_{m=1}^{M} \sum_{n=1}^{N_m} I_{mn} e^{j\Psi_{mn}(\theta, \phi)} \]

\[ AF_P(\theta, \phi) = \prod_{p=1}^{P} \left\{ \sum_{m=1}^{M} \sum_{n=1}^{N_m} I_{mn} e^{j\delta_{p-1}\Psi_{mn}(\theta, \phi)} \right\} \]
Circular Ring Array

Stage 1

Stage 2
Stages of Growth in a Binomial Array

Stage 1

Stage 2

Stage 3

Stage 4
Binomial Planar Array Generator

\[ r = \frac{\lambda}{2\sqrt{2}} \]
Stage 1

Stage 2

Stage 3
Hexagonal Arrays

A hexagon planar array geometry

A hexagon planar array can be considered as consisting of a number of concentric six-element circular arrays of different radii.
Hexagonal Arrays

\[ AF_p(\theta, \phi) = \frac{I_o + \sum_{p=1}^{P} \sum_{m=1}^{P} \sum_{n=0}^{5} I_{pmn} \exp \{ j[kr_{pm} \sin \theta \cos (\phi - \phi_{pmn}) + \alpha_{pmn}] \}}{I_o + \sum_{p=1}^{P} \sum_{m=1}^{P} \sum_{n=0}^{5} I_{pmn}} \]

where

\[ r_{pm} = d \sqrt{p^2 + (m-1)^2 - p(m-1)} \]

\[ \phi_{pmn} = \cos^{-1}\left[ \frac{r_{pm}^2 + d^2 p^2 - d^2 (m-1)^2}{2r_{pm}dp} \right] + \frac{n\pi}{3} \]

\[ \alpha_{pmn} = -kr_{mn} \sin \theta_o \cos(\phi_o - \phi_{pmn}) \]

\[ P = \text{the number of concentric hexagons in the array} \]
Hexagonal Array Generator

Hexagonal Array Generator

\[ r = \frac{\lambda}{2} \]
Hexagonal Arrays

The generating array factor for a uniformly excited six-element circular array of radius $r = \lambda / 2$ is

$$GA(\theta, \phi) = \sum_{n=1}^{6} e^{j\Psi_n(\theta, \phi)}$$

where

$$\Psi_n(\theta, \phi) = \pi [\sin \theta \cos(\phi - \phi_n) - \sin \theta_o \cos(\phi_o - \phi_n)]$$

$$\phi_n = (n - 1) \frac{\pi}{3}$$

This can be used to generate hexagonal arrays to arbitrary stage of growth $P$ with radiation patterns given by

$$AF_P(\theta, \phi) = \frac{1}{6^P} \prod_{p=1}^{P} \sum_{n=1}^{6} e^{j\delta^{p-1}\Psi_n(\theta, \phi)}$$
Generating Sub-array
Sierpinski Carpet Arrays

The generating array for a Sierpinski carpet may be represented as:

\[
\begin{array}{cccc}
1 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 1 \\
\end{array}
\]

This configuration may be interpreted as consisting of two concentric, four-element circular arrays. The corresponding expression for the Sierpinski carpet array factor is given by:

\[
AF_p(\theta, \phi) = \frac{1}{8^p} \prod_{p=1}^{P} \sum_{m=1}^{2} \sum_{n=1}^{4} e^{j3^{p-1}\Psi_{mn}(\theta, \phi)}
\]

where

\[
\Psi_{mn}(\theta, \phi) = \sqrt{m\pi}\left[\sin\theta\cos(\phi - \phi_{mn}) - \sin\theta_o\cos(\phi_o - \phi_{mn})\right]
\]

\[
\phi_{mn} = \left(\frac{mn - 1}{m}\right)\pi \times \frac{1}{2}
\]